Acoustic streaming in annular thermoacoustic prime-movers

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The theory of acoustic streaming in an annular thermoacoustic prime-mover is developed. It is predicted that above the threshold for traveling wave excitation the device considered (which does not contain any moving parts or externally imposed pressure gradients) produces circulation of fluid. The heat flux carried by this directional mass flow inside the thermoacoustic stack exceeds (or is comparable with) the heat flux associated with the acoustically induced increase of thermal diffusivity of the gas. The effects investigated are important for optimization of the performance of thermoacoustic devices. [S0001-4966(00)00309-X]

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INTRODUCTION

A thermoacoustic prime-mover is a gas-filled acoustic resonator in which a system of solid plates is installed parallel to the resonator axis (i.e., so-called stack), and which is subjected to a temperature gradient by external heating.[1] When the external thermal action on the system exceeds some critical value, gas oscillations start in the absence of any other driver (such as a loudspeaker).[1–4] Thermal energy is transformed into the energy of acoustic oscillations whose amplitude initially grows. Subsequent saturation of sound wave amplitude growth is caused by one of a number of possible nonlinear mechanisms.[2,3,5,6] Among these mechanisms is reverse influence of the acoustic waves on the temperature distribution along the stack. In accordance with Le Chatelier's principle,[7] the excitation of the acoustic waves should induce processes which lead to a reduction of the externally imposed temperature gradient. One of these processes is well understood. In the presence of a temperature gradient the acoustic oscillations will induce effective heat transport via interaction with temperature oscillations.[1] Speaking generally, we can say that in an ideal gas a nonzero enthalpy flow $\rho_m c_p \langle \upsilon_x T \rangle$ is induced[1] (where ρ_m denotes the mean density, c_p is the isobaric heat capacity, υ_x denotes the axial component of the oscillating particle velocity in the acoustic wave ($\langle \upsilon_x \rangle = 0$), and T is the oscillating component of the temperature field ($\langle T \rangle = 0$), $\langle ... \rangle$ standing for time averaging over a period of the oscillation). In the case of a thermoacoustic prime-mover, this flow can cause smoothing of the externally imposed temperature gradient.[2,8,9] Another process which is expected to be induced by the oscillations and which is expected to contribute to heat transport[1] is the excitation of acoustic streaming.[10] A nonzero mass flow $M \equiv (\rho_m \upsilon_{xm} + \langle \rho \upsilon_x \rangle)$ is generated in the acoustic field as a consequence of various nonlinear processes[10,11] [here, υ_{xm} denotes the mean axial component of the particle velocity, ρ is the oscillating component of the density ($\langle \rho \rangle = 0$)]. The role of the associated enthalpy flow $M c_p T_m \equiv \rho_m c_p [\upsilon_{xm} + (\langle \rho \upsilon_x \rangle / \rho_m)] T_m$ (where T_m is the mean temperature) is much less understood in thermoacoustic theory. Even an order-of-magnitude estimate of the relative importance of the two heat-transfer mechanisms described above (i.e., an estimate of the ratio $M T_m / (\rho_m \langle \upsilon_x T \rangle)$, for example) is not available in the literature.

The first experimental observations of the acoustic streaming in a standing wave thermoacoustic device were reported in Ref. 12. A photograph of acoustic streaming in a standing-wave thermoacoustic prime-mover is presented in Ref. 13. Recently, the importance of acoustic streaming in the operation of a pulse-tube refrigerator was demonstrated experimentally.[14] It should be noted that in these experimental configurations[12–14] the total time-averaged mass flow through any cross section of the resonator (and, consequently, through any cross section of thermoacoustic stack) is equal to zero. As the net mass flow along the tube must be zero, then in each cross section of the resonator there exist upward and downward streaming currents.[10,14] In general, this situation is expected to diminish the role of the acoustic streaming in the heat transport along the stack in comparison with the devices where there exists a closed-loop path for steady streaming and where, as a consequence, a nonzero net mass flow is possible. A discussion of how a nonzero net time-averaged mass flow can arise in Stirling and pulse-tube cryocoolers whenever a closed-loop path exists for steady flow can be found in Ref. 15. Recent experiments[16] confirm the existence of these circular (closed-loop) streamings. Moreover, it was demonstrated[16] that suppression of the streaming leads to a significant increase in the efficiency of the device performance. This observation confirms the extremely important role of heat transfer by acoustic streaming in pulse-tube refrigerators (with an existing closed loop for hydrodynamic streaming). It also leads to the conclusion[16] that in the traveling wave thermoacoustic devices (where there exists a closed-loop path for a unidirectional acoustic wave, and, consequently, there is a closed-loop path for a nonzero mass flow)[17] the role of the acoustic streaming could be very important as well. Recently, a traveling wave ther-

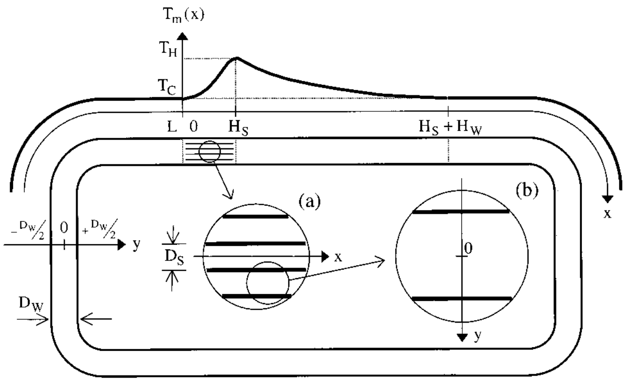

FIG. 1. Schematic presentation of an annular thermoacoustic prime-mover and the gas temperature distribution $T_m(x)$ along its axis (x axis). Here, L denotes the total length of the annular resonator $(0 \leqslant x \leqslant L), H_S$ is the length of thermoacoustic stack, H_W is the length of region with decreasing temperature $(dT_m/dx < 0)$, and D_W is the width of the waveguide. The inset (a) presents the structure of the stack with plate separation denoted by D_S. The inset (b) presents the coordinate axis y which is used in the theoretical analysis of the thermoacoustic phenomena inside the stack. The comparable coordinate axis for the waveguide part of the resonator is shown on the left-hand side of the figure.

moacoustic prime-mover was created for the first time in practice.[18] The first experimental investigations of a role of the acoustic streaming in operation of an annular thermoacoustic prime-mover has been reported in Ref. 19. These experimental observations[17–19] make it of interest to develop the theory for acoustic streaming in annular thermoacoustic devices.

The present publication addresses the theoretical analysis of acoustic streaming in annular thermoacoustic prime-movers. The schematic presentation of the device together with a qualitative picture of mean temperature $T_m(x)$ distribution along the x axis of the device is given in Fig. 1. It is assumed that the system is above threshold for thermoacoustic instability. The following specific features of this resonator system are taken into account in the analysis of streaming excitation: (1) the existence of spatial regions with very different resistance to hydrodynamic flow (i.e., the stack and the rest of the resonator), (2) the existence of regions with spatially inhomogeneous properties [for example, both the temperature $T_m(x)$ and the density $\rho_m \propto 1/T_m(x)$ are x-dependent inside the region $0 \leqslant x \leqslant H_S + H_W$ in Fig. 1], (3) the existence of regions where the acoustic wave is amplified and regions where it is attenuated (i.e., regions of growing and diminishing wave amplitude). Particular attention is paid to identification of the sources of acoustic streaming. It is demonstrated that volume Reynolds stresses, which can be presented as an x derivative of a potential [for example, $\langle v_x \partial v_x/\partial x \rangle = (1/2)\partial \langle v_x^2 \rangle/\partial x$] and which can play a major role in systems where the path of the acoustic wave and the path of streaming do not coincide completely,[20] are not active in streaming excitation in annular devices. These volume sources are totally compensated by pressure gradients because of the x periodicity of both the acoustic wave and the streaming. On the other hand, such volume sources of streaming as, for example, $\propto \partial \langle v_x v_y \rangle/\partial y$ [where v_y denotes oscillating part of the y component of particle velocity $(\langle v_y \rangle = 0)$ and y is the lateral coordinate in Fig. 1], which do not contribute to flow excitation in a plane wave approximation[10,20,21] $(v_y \equiv 0)$, can be important in traveling wave devices.

In Sec. I we formulate all the basic assumptions which are necessary to make the problem tractable without losing important physical features of the phenomena. In Sec. II a general solution for the cross-sectional average mass flow in an annular thermoacoustic device is obtained. Section III is devoted to a detailed analysis of the mass flow and the acoustic streaming velocity in the case where the interaction between the acoustic waves and the stack is quasiadiabatic. In Sec. IV the limiting regime of the quasi-isothermal interaction between the sound and the stack is evaluated. Section V presents the results of a comparison of the magnitude of the streaming-induced enthalpy flow with the enthalpy flow associated with the additional acoustically induced thermal conductivity of the gas. This is followed by conclusions.

I. THEORETICAL ASSUMPTIONS

The goal of the present analysis is to gain insight into the physics of the streaming phenomenon in thermoacoustic devices and to derive theoretical relationships which provide reliable order-of-magnitude estimates for the effects considered. To achieve this in the simplest way, a few assumptions concerning the geometry of the resonator and the stack are made. The assumptions concerning the structure of an annular thermoacoustic prime-mover are the following:

(1) The looped tube in Fig. 1 has a rectangular rather than a circular cross section, and we analyze it as a two-dimensional problem neglecting the dependence of the physical quantities on the z coordinate (which is orthogonal to the plane of Fig. 1). In other words, we consider that the annular resonator is composed of two wave-guiding surfaces parallel to the z axis. This assumption provides for a more compact presentation of some of the results in comparison with the case of a cylindrical tube. However, it should be noted that the results obtained can be used for order-of-magnitude estimates for cylindrical tubes as well.

(2) The looped waveguide has an interwall distance D_W which is significantly less than the total length L of the waveguide (W)

$$L \gg D_W. \tag{1}$$

This condition allows us to neglect the influence of channel curvature (i.e., of waveguide looping) on the propagation of sound waves and on the hydrodynamic flow. Because of this assumption the looped coordinate x in Fig. 1 can be straightened in the subsequent analysis. The looping will be taken into account in this paper by imposing periodicity conditions on the physical quantities $\psi(x=L) = \psi(x=0)$ (where ψ stands for an arbitrary physical function of the problem under consideration).

(3) The external action on the system is achieved by the stack (S) in the region $0 \leqslant x \leqslant H_S$ in Fig. 1 and two heat exchangers near its edges (to cool the left edge and to heat the right edge of the stack). The axial dimensions of

the heat exchangers are significantly less than the length, H_S, of the stack. Their influence on the acoustic wave and their hydrodynamic resistance to acoustic streaming are neglected in the analysis (and consequently they are not presented in Fig. 1). In order to achieve good coupling between thermal and acoustic waves, the stack should contain many channels.[1] Consequently, we assume that the width, D_S, of individual channels in the stack is much less than the width of the waveguide

$$D_W \gg D_S. \tag{2}$$

For simplicity we neglect possible blockage of the sound waves and of the streaming by the stack (i.e., we assume that the stack has a porosity equal to 1).

(4) The length of the waveguide is much greater than the stack length, i.e.,

$$L \gg H_S. \tag{3}$$

This corresponds to practical experimental configurations ($L/H_S > 60$ in Ref. 18). Importantly, for the fundamental acoustic resonance (where the acoustic wavelength λ is similar to the resonator length L), Eq. (3) is equivalent to the statement that thermoacoustic stack is acoustically thin. As a consequence, the inequality (3) allows us a simplified analysis of acoustic wave propagation in the stack where the medium is spatially inhomogeneous due to heating and where the traditional representation of the acoustic field as the superposition of two counterpropagating waves is generally invalid.[22]

(5) $$(L/H_S)(D_S/D_W)^2 \ll 1. \tag{4}$$

This relates assumptions (2) and (3) and indicates that the stack provides an important increase in the area of boundary layers (where the interaction between the acoustic and thermal waves mostly takes place) without occupying a significant amount of the resonator volume.

(6) The final assumption concerning the stack dimensions is

$$H_S \gg D_S, \tag{5}$$

which is fulfilled in all thermoacoustic devices.[1–4,18]

(7) The region $0 \leq x \leq H_S$ of the temperature increase is accompanied by another region with spatially inhomogeneous properties where temperature gradually falls to its ambient value T_C. The characteristic length of this region is denoted by H_W in Fig. 1. In the region $H_S \leq x \leq H_S + H_W$, the lowering of temperature is caused by heat loss from the gas to the walls of the waveguide. The length H_W can be also fixed by installation of an additional heat exchanger at ambient temperature in the position $x = H_S + H_W$. The following assumptions are made concerning the scale of H_W:

$$L \gg H_W, \tag{6}$$

$$H_W \gg D_W. \tag{7}$$

These are similar to the assumptions expressed in Eq. (3) and Eq. (5), respectively. Note that Eq. (1) follows from Eq. (6) and Eq. (7).

II. GENERAL SOLUTION FOR THE ACOUSTIC STREAMING

The equation describing the lateral variation of the axial streaming velocity v_{xm} in the boundary layer in the presence of an axial temperature gradient was derived by N. Rott[11]

$$\nu_m \frac{\partial^2}{\partial y^2} v_{xm} = \frac{1}{\rho_m} \frac{\partial}{\partial x} (p_h + \rho_m \langle v_x^2 \rangle) + \frac{\partial}{\partial y} (\langle v_x v_y \rangle) - \beta \frac{\nu_m}{T_m} \frac{\partial}{\partial y} \left(\left\langle T \frac{\partial}{\partial y} v_x \right\rangle \right). \tag{8}$$

Here, y is the coordinate normal to the wall (Fig. 1), ν_m is the fluid viscosity evaluated at the mean temperature ($\nu_m \propto T_m^{\beta+1}$), and p_h is the hydrodynamic pressure (which accompanies the streaming). Note that the last term on the right-hand side (rhs) of Eq. (8) describes the excitation of streaming due to the dependence of fluid viscosity on temperature.[11,14] The hydrodynamic pressure p_h does not depend on the y coordinate.

Equation (8) is based on the boundary layer equation and was supposed valid only in the boundary layer.[11] However, the streaming circulating in the annular device has, in fact, the form of a jet (bounded by the plates) and the hydrodynamic equations for a jet are known to have the same form as those for the boundary layer flow[23] (because the important scaling property $\partial/\partial x \ll \partial/\partial y$ is the same in both cases). In the prime-mover we are considering, the scaling property $\partial/\partial x \ll \partial/\partial y$ is effected by the conditions (1), (5), and (7). Consequently, Eq. (8) can be applied to the evaluation of the streaming velocity in the dominant part of the annular device.

The terms which are neglected in Eq. (8) should be taken into account and the Navier–Stokes equation applied for the lateral component v_{ym} of streaming velocity near the cross sections located at $x=0$, $x=H_S$, and $x=H_S+H_W$, but only if a description giving a smooth transformation of velocity profile in the vicinity of the stack edges and at the boundary between the heated and cold part of the resonator is desirable. However, the most important features of acoustic streaming in annular thermoacoustic prime-movers can be described, when matching the flows at the interfaces between different parts of the device, by using the conditions for the physical quantities averaged over the cross-section of the waveguide. For the present analysis, the most useful of these conditions is continuity of the cross-sectional average mass flow $\overline{M}$

$$\overline{M} \equiv \rho_m \overline{v_{xm}} + \overline{\langle \rho v_x \rangle} \equiv \text{const}. \tag{9}$$

Here, the procedure of averaging over the cross section is defined by $\overline{(\cdots)} \equiv (1/2)\int_{-1}^{1}(\cdots)d\eta$, where the dimensionless coordinate $\eta = 2y/D$ is introduced [$D=D(x)$ is equal to the separation D_W of the walls in the waveguide or equal to the separation D_S of the plates in the stack]. Relationship (9) can be proved by integrating the time-averaged continuity equation $\partial(\rho_m v_{xm} + \langle \rho v_x \rangle)/\partial x + \partial(\rho_m v_{ym} + \langle \rho v_y \rangle)/\partial y = 0$ over the cross section of the channel. Applying the boundary condition that at the walls of each individual channel $v_{ym} = v_y = 0$, we derive $\partial(\rho_m \overline{v_{xm}} + \overline{\langle \rho v_x \rangle})/\partial x = \partial \overline{M}/\partial x = 0$, and, finally, we get Eq. (9). It describes a rather evident physical fact that under stationary conditions the mean mass flow is the same in the stack and in the waveguide.

The steps in the evaluation of Eq. (8) are: (1) integrate twice over the η coordinate, taking into account the conditions $v_y(\eta=0)=0$, $\partial v_x/\partial\eta(\eta=0)=0$, $v_{xm}(\eta=-1)=0$, and the nondependence of T_m on η; (2) average over the cross section. The result is

$$\frac{\nu_m}{(D/2)^2}\overline{v_{xm}} = \frac{1}{\rho_m}\frac{\partial}{\partial x}\left(\overline{\int_{-1}^{\eta} d\eta' \int_0^{\eta'} d\eta''(p_h+\rho_m\langle v_x^2\rangle)}\right) + \frac{1}{(D/2)}\overline{\int_{-1}^{\eta} d\eta'\left(\langle v_x v_y\rangle - \frac{1}{(D/2)}\beta\frac{\nu_m}{T_m}\left\langle T\frac{\partial}{\partial\eta'}v_x\right\rangle\right)}. \tag{10}$$

After the substitution of $\overline{v_{xm}}$ from Eq. (10) into Eq. (9), it is appropriate to present the result in the form

$$\frac{\nu_m}{(D/2)^2}\overline{M} = \frac{\partial}{\partial x}\left(\overline{\int_{-1}^{\eta} d\eta' \int_0^{\eta'} d\eta''(p_h+\rho_m\langle v_x^2\rangle)}\right) + \frac{\rho_m}{(D/2)} \times \overline{\int_{-1}^{\eta} d\eta'\left(\langle v_x v_y\rangle - \frac{1}{(D/2)}\beta\frac{\nu_m}{T_m}\left\langle T\frac{\partial}{\partial\eta'}v_x\right\rangle\right)} + \frac{\nu_m}{(D/2)^2}\langle\overline{\rho v_x}\rangle. \tag{11}$$

Now, the integration of Eq. (11) with respect to x over the closed loop $(0\leqslant x\leqslant L)$ provides an opportunity to eliminate the function

$$\overline{\int_{-1}^{\eta} d\eta' \int_0^{\eta'} d\eta''(p_h+\rho_m\langle v_x^2\rangle)}$$

which includes the unknown pressure distribution p_h. This is possible because of the spatial periodicity of the considered system (the period is equal to the total length L of the annular resonator). Taking into account that [in accordance with Eq. (9)] $\overline{M}$ on the left-hand side of Eq. (11) does not depend on the x coordinate, we obtain

$$\overline{M} = \oint s(x)dx \Big/ \oint (2/D)^2\nu_m\, dx. \tag{12}$$

Here, we call

$$s\equiv\overline{\frac{2\rho_m}{D}\int_{-1}^{\eta} d\eta'\left(\langle v_x v_y\rangle - \frac{2\beta}{D}\frac{\nu_m}{T_m}\left\langle T\frac{\partial v_x}{\partial\eta'}\right\rangle\right) + \frac{4\nu_m}{D^2}\langle\rho v_x\rangle} \tag{13}$$

the density of sources inducing the acoustic streaming. When integrated over x between two cross sections of the resonator, the introduced density of the mass flow sources provides the effective difference in Reynolds stresses between these cross sections. Consequently, the denominator in Eq. (12) can be called the hydrodynamic resistance to the mass flow. If the acoustic field is known, then Eqs. (12) and (13) provide a solution for the circulating mass flow. If the mass flow $\overline{M}$ is found, then the average streaming velocity $\overline{v_{xm}}$ can be evaluated in each particular part of the annular thermoacoustic prime-mover with the help of Eq. (9)

$$\overline{v_{xm}} = \frac{1}{\rho_m}(\overline{M} - \langle\overline{\rho v_x}\rangle). \tag{14}$$

In accordance with Eq. (14), $\overline{v_{xm}}$ (in contrast to $\overline{M}$) depends on the axial coordinate x.

The equation for the steady-state pressure oscillations p in thermoacoustic devices is well-known.[1,24] It is presented here for an ideal gas, neglecting the temperature oscillations in the walls of the tube and in the plates composing the stack

$$\frac{d^2\tilde{p}}{dx^2} + \left\{1+\frac{1}{1-f_\nu}\left[\frac{f_\nu-f_k}{1-\sigma} - T_N\frac{\partial f_\nu}{\partial T_N}\right]\right\}\frac{1}{T_N}\frac{dT_N}{dx}\frac{d\tilde{p}}{dx} + k_c^2\left\{1+\frac{1}{1-f_\nu}[f_\nu+(\gamma-1)f_k]\right\}\frac{1}{T_N}\tilde{p}=0\,. \tag{15}$$

In Eq. (15) and later on we use the notation $\tilde{\psi}$ for the complex amplitude of the harmonic function ψ, in particular for the pressure oscillations $p\equiv\mathrm{Re}[\tilde{p}\exp(-i\omega t)]$, where ω denotes the cyclic frequency of the oscillations and t is the time. In Eq. (15) T_N denotes the mean temperature normalized to the temperature T_c in the cold homogeneous part of the resonator $(T_N=T_m/T_c)$, $k_c=\omega/a_c$ is the adiabatic acoustic wave number in the cold gas in the absence of the solid boundaries, $a_c=\sqrt{\gamma p_m/\rho_c}$ is the adiabatic sound velocity in the cold part of the resonator, γ is the ratio of specific heats, p_m is the mean pressure, and $\rho_c\equiv\rho_m(T_m=T_c)$.

The functions

$$f_\nu\equiv\frac{(1+i)\delta_\nu}{D}\tanh\left[\frac{D}{(1+i)\delta_\nu}\right], \quad f_k\equiv\frac{(1+i)\delta_k}{D}\tanh\left[\frac{D}{(1+i)\delta_k}\right], \tag{16}$$

characterize the interaction between the acoustic wave and the solid boundaries. In Eq. (16) the width of the viscous boundary layer $\delta_\nu\equiv\sqrt{2\nu_m/\omega}$ and the width of the thermal boundary layer $\delta_k\equiv\sqrt{2k_m/\omega}$ (where k_m is the thermal diffusivity of the gas at mean temperature) depend on the mean temperature because $\nu_m\propto k_m\propto T_m^{\beta+1}$ (β is the phenomenological parameter[25,26]). Consequently, the functions f_ν and f_k depend on the x coordinate. Finally, in Eq. (15) $\sigma\equiv\nu_m/k_m$ denotes the Prandtl number.

It should be mentioned that there is an additional assumption for the validity of Eq. (15) in thermoacoustic prime-movers. In the case of the thermoacoustic prime-movers, it is implicitly assumed that the stabilization (saturation of the growth) of the wave amplitude, when the system is above the threshold of thermoacoustic instability, is not caused by the nonlinear effects in the acoustic wave propagation. The corresponding nonlinear terms (see, for example, Refs. 3, 6, 27–30) are neglected in Eq. (15). The physical process leading to the stabilization of the acoustic wave amplitude in this case can be described as follows. (1) If the temperature distribution induced in the stack by an external heating is sufficiently steep and the amplification of the acoustic wave inside the stack exceeds its attenuation in the rest of the resonator, the amplitude $|\tilde{p}|$ of the pressure oscillations starts to grow. (2) The rise of the amplitude of the acoustic oscillations increases the acoustically induced thermal conduction[6,8,9,31,32] and the acoustically induced mass flow.[15,16,19] Both these effects (proportional to $|\tilde{p}|^2$ to a first approximation) cause the nonlinear increase in the heat transport between the limits $x=0$ and $x=H_S$ of the thermoacoustic stack. (3) In the case when the external thermal action on the system is fixed (if, for example, the total heat flux supplied to the hot end of the stack is fixed), the increased acoustically induced heat transport through the gas induces a smoothing of the temperature gradient across the stack and a reduction of the sound amplification. (4) The growth of the amplitude of the pressure oscillations and the smoothing of the temperature distribution compensate each other when the total amplification of the acoustic waves in the prime-mover becomes equal to zero. At this point, the steady-state equation (15) is valid.

Consequently, in the case of the thermoacoustic prime-mover, the temperature distribution in Eq. (15) itself depends, in fact, on the acoustic field. For example, in order to find the amplitude of the stationary wave in the system, the equation for the temperature distribution (which includes acoustically induced heat transport) should be added to Eq. (15). In order to study the stability of the stationary oscillations, these coupled equations should be modified to include the transient processes (for example, the process of the acoustic streaming development[10,20,33]). However, both these physical problems are far beyond the scope of the present paper (though they provide clear a perspective for future extensions of the theory).

For the analysis presented below, it is important that Eq. (15) can be used to derive some general conclusions concerning the acoustic streaming even without finding a specific solution. In fact, the solutions for v_x, v_y, T, and ρ in the lowest-order acoustic mode of the resonator (i.e., in the mode which becomes a homogeneous plane wave if the viscosity and the thermal conductivity are negligible) are the known functions of p distribution[24] (see Appendix A). So, we are able to present the density of sources $s(x)$ as a function of $\tilde{p}$, $d\tilde{p}/dx$, and $d^2\tilde{p}/dx^2$, and to estimate the sources of the mass flow in the different parts of the annular device. It is possible to estimate the mass flow and to compare the enthalpy fluxes carried by the directional mass flow and by the acoustically induced thermal conduction.

Using the results (A1)–(A4) presented in Appendix A and the relation $\langle gh\rangle=(1/2)\mathrm{Re}(\tilde{g}\tilde{h}^*)$ (where * denotes the complex conjugate, and g, h are the arbitrary functions), we are able to find all time-averaged nonlinear terms contributing to Eq. (13). For the sources of the mass flow we derive the following general formula:

$$
\begin{aligned}
s=&-\frac{1}{2\omega^2\rho_m}\mathrm{Re}\Bigg\{\frac{1}{T_m}\frac{dT_m}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\overline{\int_{-1}^{\eta}d\eta'\Bigg\{\left(\frac{\beta+1}{2}\right)\Bigg[\eta' F_\nu-\left(1+\left(\frac{D}{(1+i)\delta_\nu}\right)^2 f_\nu\right)\Phi_\nu-\eta' F_\nu F_\nu^*+\left(1+\left(\frac{D}{(1+i)\delta_\nu}\right)^2 f_\nu\right)\Phi_\nu F_\nu^*\Bigg]}\\
&\overline{+\left[\eta'-\frac{\sigma}{\sigma-1}\Phi_\nu+\frac{1}{\sigma-1}\Phi_k-\eta' F_\nu^*+\frac{\sigma}{\sigma-1}\Phi_\nu F_\nu^*-\frac{1}{\sigma-1}\Phi_k F_\nu^*\right]-\beta\left[\Phi_\nu^*-\frac{\sigma}{\sigma-1}F_\nu\Phi_\nu^*+\frac{1}{\sigma-1}F_k\Phi_\nu^*\right]}\\
&+\frac{d^2\tilde{p}}{dx^2}\frac{d\tilde{p}^*}{dx}\overline{\int_{-1}^{\eta}d\eta'(\eta'-\Phi_\nu-\eta' F_\nu^*+\Phi_\nu F_\nu^*)}\\
&+\left(\frac{\omega}{a}\right)^2\tilde{p}\frac{d\tilde{p}^*}{dx}\overline{\int_{-1}^{\eta}d\eta'[\eta'-\eta' F_\nu^*+(\gamma-1)\Phi_k-(\gamma-1)\Phi_k F_\nu^*+\beta(\gamma-1)\Phi_\nu^*-\beta(\gamma-1)F_k\Phi_\nu^*]}\Bigg\}\\
&+\frac{1}{2\omega^2\rho_m}\mathrm{Re}\Bigg\{\frac{1}{T_m}\frac{dT_m}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\frac{1}{(\sigma-1)}\left[\frac{(1+i)\delta_\nu}{D}\right]^2\overline{(F_k+F_\nu^*-F_k F_\nu^*)}\\
&+\left(\frac{\omega}{a}\right)^2\tilde{p}\frac{d\tilde{p}^*}{dx}\left[\frac{(1+i)\delta_\nu}{D}\right]^2\overline{[1-F_\nu^*+(\gamma-1)F_\nu-(\gamma-1)F_k F_\nu^*]}\Bigg\}. \qquad (17)
\end{aligned}
$$

In Eq. (17), the last term in curly brackets (which does not contain any other integration except the averaging over the cross section of the channel) describes the contribution $(4\nu_m/D^2)\langle\overline{\rho v_x}\rangle$ to the sources s. We have not done any significant regrouping of the terms in Eq. (17) with the intention of facilitating for the reader a verification of the solution if desired. The only rearrangement during the derivation of Eq. (17) is the separation of the terms proportional to $(1/T_m)(dT_m/dx)|d\tilde{p}/dx|^2$, to $(d^2\tilde{p}/dx^2)(d\tilde{p}^*/dx)$, and to $(\omega/a)^2\tilde{p}(d\tilde{p}^*/dx)$. The reason for this procedure will become clear later.

All the spatial integrals necessary for the evaluation of Eq. (17) are presented in Appendix B. Consequently, Eqs. (12) and (17), together with the results presented in the Appendix B, provide the description of the circulating mass flow in annular thermoacoustic prime-movers in terms of the temperature distribution and the acoustic pressure distribution inside the resonator.

For example, with the help of Eqs. (B7) and (B8) the solution for the component $\langle \overline{\rho v_x} \rangle$ of the total mass flow can be presented in a compact form

$$\langle \overline{\rho v_x} \rangle = -\frac{1}{2\omega^3 \rho_m} \left\{ \frac{1}{(\sigma^2-1)} \frac{1}{T_m} \frac{dT_m}{dx} \left| \frac{d\tilde{p}}{dx} \right|^2 \mathrm{Im}(f_k + \sigma f_\nu^*) + \left(\frac{\omega}{a} \right)^2 \mathrm{Im}\left[\left(1 - f_\nu^* + \frac{\gamma-1}{\sigma+1}(f_k - f_\nu^*) \right) \tilde{p} \frac{d\tilde{p}^*}{dx} \right] \right\}. \tag{18}$$

As we have described earlier, the distribution of the mean temperature in the system and the acoustic field themselves depend on the mass flow $\overline{M}$ because $\overline{M}$ contributes to the enthalpy transport in the system. So, the derived description obtained in Eqs. (12) and (17) is not, in fact, a solution for the mass flow but just one of the equations in the set describing the thermoacoustic processes in the system. In general, these equations should be solved simultaneously with the equation for the heat transport in the thermoacoustic prime-mover. However, it appears that using Eqs. (12) and (17) we can reach conclusions concerning the direction of the mass flow and the acoustic streaming velocity in the system without needing to solve the general problem.

For this purpose we analyze separately the limiting cases of the quasiadiabatic and of the quasi-isothermal interaction of the acoustic and thermal waves inside the stack $(0 \leqslant x \leqslant H_S)$. Note that in both cases under consideration the thermoacoustic interactions in the rest of the resonator are assumed to be quasiadiabatic (in accordance with all known experimental configurations[9,19]).

III. QUASIADIABATIC REGIME

In the quasiadiabatic (QA) regime of the interaction between the acoustic waves and the solid surfaces (which is characterized by the fact that the widths of the viscous and thermal boundary layers are significantly narrower than the distance D between the channel walls, i.e., $\delta_{\nu,k} \ll D_{S,W}$), there is a small parameter in the theory

$$|f_\nu| \approx \sqrt{\sigma}|f_k| \approx \left| \frac{(1+i)\delta_\nu(x)}{D(x)} \right| \equiv \frac{\sqrt{2}\delta_\nu}{D} \propto \frac{\delta_\nu}{D} \ll 1. \tag{19}$$

Retaining in Eq. (15) only those terms not smaller than the first order in the parameter $|f_\nu| \ll 1$, we get

$$\frac{d^2\tilde{p}}{dx^2} + \left[1 - \frac{(\beta+1)}{2}(2c+d)f_\nu \right] \frac{1}{T_N} \frac{dT_N}{dx} \frac{d\tilde{p}}{dx} + k_c^2(1+2cf_\nu)\frac{1}{T_N}\tilde{p} = 0. \tag{20}$$

Here, $c \equiv (1/2)[1+(\gamma-1)/\sqrt{\sigma}]$ is Kirchhoff's constant and $d \equiv (1/\sqrt{\sigma})[2(1+\beta)^{-1}(1+\sqrt{\sigma})^{-1}-(\gamma-1)]$ is Kramer's constant.[25,26]

In the cold part of the resonator, i.e., where $H_S + H_W \leqslant x \leqslant L$, $dT_N/dx \equiv 0$ $(T_N \equiv 1)$ and hence Eq. (20) is additionally simplified

$$\frac{d^2\tilde{p}}{dx^2} + k_c^2(1+2cf_\nu)\tilde{p} = 0. \tag{21}$$

In this cold part of the resonator the most important (leading) contributions both to $\langle \overline{\rho v_x} \rangle$ and to $\overline{M}$ can be found by retaining in Eqs. (18) and (17) only those terms of zero order in the small parameter $|f_\nu| \ll 1$.

$$\langle \overline{\rho v_x} \rangle_c^{\mathrm{QA}} \cong -\frac{1}{2\omega\rho_c a_c^2} \mathrm{Im}\left(\tilde{p} \frac{d\tilde{p}^*}{dx} \right) \cong \frac{1}{2\rho_c a_c^3} |\tilde{p}|^2, \tag{22}$$

where the relation between $d\tilde{p}/dx$ and $\tilde{p}$ is approximated by the form it has in a wave traveling in the positive x direction

$$d\tilde{p}/dx \approx ik_c\tilde{p}. \tag{23}$$

Note that we are using the suffix c ("cold") for functions and parameters in the homogeneous region of the device $(H_S + H_W \leqslant x \leqslant L)$. The description of the density of the mass flow sources following from Eq. (17) for the cold region is

$$s_c^{\mathrm{QA}} \cong \frac{1}{6\omega^2\rho_c} \mathrm{Re}\left[\frac{d^2\tilde{p}}{dx^2} \frac{d\tilde{p}^*}{dx} + \left(\frac{\omega}{a_c} \right)^2 \tilde{p} \frac{d\tilde{p}^*}{dx} \right]. \tag{24}$$

With the help of Eq. (21) and then Eq. (23) [taking into account the inequality (19)], we obtain the density of the sources of the mass flow in the cold part of the device Eq. (24) in the final form

$$s_c^{\mathrm{QA}} \cong -\frac{c\omega}{3\rho_c a_c^3} \left(\frac{\delta_{\nu c}}{D_W} \right) |\tilde{p}|^2, \tag{25}$$

where $\delta_{\nu c}$ is the thickness of the viscous boundary layer in the cold part of the resonator. Note that the effective sources in the cold part of the resonator are directed in a direction opposite to that of the traveling wave.

For the estimates in the heated part $(0 \leqslant x \leqslant H_S + H_W)$ of the thermoacoustic device, we adopt a strategy based on the following assumptions. First, we assume that the heated region is acoustically thin

$$(H_S + H_W)/\lambda \ll 1, \tag{26}$$

where λ is the acoustic wavelength inside the stack. Second, because the heated region should provide the amplification of the acoustic wave, we assume for Eq. (20) that the term related to the temperature gradient (the second term) dominates over the last term. Thus, we approximate Eq. (20) in the inhomogeneously heated region by

$$\frac{d^2\tilde{p}}{dx^2} + \left[1 - \frac{(\beta+1)}{2}(2c+d)f_\nu \right] \frac{1}{T_N} \frac{dT_N}{dx} \frac{d\tilde{p}}{dx} \cong 0. \tag{27}$$

To get a rough condition for the validity of the transition from Eq. (20) to Eq. (27) [i.e., for neglecting terms proportional to $(\omega/a)^2 \propto 1/\lambda^2$], we can approximate $d\tilde{p}/dx$ in the acoustically thin heated region by its value at the stack edge (at $x=0$, for example). The latter can be found from the condition of continuity of the average velocity $\overline{\tilde{v}_x}$ at $x=0$

$[\overline{\tilde{v}_x}(x=0-0)=\overline{\tilde{v}_x}(x=0+0)]$. This condition can be presented as

$$\frac{1}{i\omega\rho_c}(1-f_\nu^W)\frac{d\tilde{p}}{dx}(0-0)=\frac{1}{i\omega\rho_c}(1-f_\nu^S)\frac{d\tilde{p}}{dx}(0+0). \quad (28)$$

We see then that Eqs. (28) and (23) do indeed show that the relation $d\tilde{p}/dx\approx ik_c\tilde{p}$ [Eq. (23)] holds in the first approximation in a QA stack and hence the ratio of the second and the third terms in Eq. (20) can be estimated as $\propto(dT_N/dx)/k_c$. Substituting the crude estimate $(dT_N/dx)\propto[(T_H-T_c)/T_c]/H\equiv(\Delta T/T_c)/H(H\equiv H_{S,W})$, we come to the conclusion that the approximate equation (27) is valid if

$$(\Delta T/T_c)\geqslant k_cH\equiv 2\pi(H/\lambda_c). \quad (29)$$

This inequality holds in thermoacoustic prime-movers because in the experiments $\Delta T\geqslant T_c$ above the threshold[18] and the inhomogeneously heated region is acoustically thin [see Eqs. (3), (6), and (26)].

When condition (29) is satisfied we can also neglect the terms $\propto(\omega/a)^2\propto 1/\lambda^2$ in Eqs. (17) and (18) when evaluating $\langle\overline{\rho v_x}\rangle_h$ and s_h^{QA}. Here, we introduced the suffix h (''heated'') for functions in the heated part of the thermoacoustic device. From Eq. (18)

$$\begin{aligned}\langle\overline{\rho v_x}\rangle_h^{\mathrm{QA}}&\cong\frac{1}{2\omega^3\rho_m}\frac{\sigma+\sqrt{\sigma}+1}{\sqrt{\sigma}(\sqrt{\sigma}+1)(\sigma+1)}\left(\frac{\delta_\nu}{D}\right)\frac{1}{T_N}\left(\frac{dT_N}{dx}\right)\left|\frac{d\tilde{p}}{dx}\right|^2\\&\cong\frac{1}{2\omega\rho_ca_c^2}\frac{\sigma+\sqrt{\sigma}+1}{\sqrt{\sigma}(\sqrt{\sigma}+1)(\sigma+1)}\left(\frac{\delta_\nu}{D}\right)\left(\frac{dT_N}{dx}\right)|\tilde{p}|^2\\&\propto\mathrm{sign}\left(\frac{dT_N}{dx}\right)\frac{1}{4\pi\rho_ca_c^3}\left(\frac{\delta_{\nu c}}{D}\right)\left(\frac{\lambda_c}{H}\right)|\tilde{p}|^2.\end{aligned} \quad (30)$$

The final order-of-magnitude estimate in Eq. (30), obtained making the approximations $\sigma\propto T_N\propto(\Delta T/T_c)\propto 1$, $dT_N/dx\propto(\Delta T/T_c)/H\approx\mathrm{const}$, is the most rough.

To find the density of the sources of the mass flow, we derive from Eq. (17)

$$s_h^{\mathrm{QA}}\cong\frac{1}{6\omega^2\rho_m}\mathrm{Re}\left(\frac{1}{T_m}\frac{dT_m}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2+\frac{d^2\tilde{p}}{dx^2}\frac{d\tilde{p}^*}{dx}\right). \quad (31)$$

Note that it was sufficient to retain in Eq. (31) only those terms of zero order in the small parameter $|f_\nu|\ll 1$. First with the aid of Eq. (27), and then using Eq. (23), we transform Eq. (31) into

$$\begin{aligned}s_h^{\mathrm{QA}}&\cong\frac{(c+d/2)}{3\omega^2\rho_m}\left(\frac{\beta+1}{2}\right)\left(\frac{\delta_\nu}{D}\right)\frac{1}{T_m}\frac{dT_m}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\\&\cong\frac{(c+d/2)}{3\rho_ca_c^2}\left(\frac{\beta+1}{2}\right)\left(\frac{\delta_\nu}{D}\right)\frac{dT_N}{dx}|\tilde{p}|^2\\&\propto\mathrm{sign}\left(\frac{dT_N}{dx}\right)\frac{1}{3\rho_ca_c^2}\left(\frac{\delta_{\nu c}}{D}\right)\frac{1}{H}|\tilde{p}|^2.\end{aligned} \quad (32)$$

In the final order-of-magnitude estimate, we have assumed that $c\propto\beta\propto T_N\propto(\Delta T/T_c)\propto 1$ and $|d/2c|\ll 1$.

The derived expressions (25) and (32) allow us to find the source of the mass flow in the nominator of Eq. (12) in the case of the quasiadiabatic stack (note that the waveguide is always quasiadiabatic)

$$\begin{aligned}S^{\mathrm{QA}}&\equiv\oint s^{\mathrm{QA}}\,dx\\&=\int_0^{H_S}s_h^{\mathrm{QA}}(D=D_S)dx+\int_{H_S}^{H_S+H_W}s_h^{\mathrm{QA}}(D=D_W)dx\\&\quad+\int_{H_S+H_W}^{L}s_c^{\mathrm{QA}}(D=D_W)dx\\&\cong\frac{|\tilde{p}|^2}{3\rho_ca_c^2}\left\{\left(c+\frac{d}{2}\right)\left(\frac{\beta+1}{\beta-1}\right)\left[\frac{\delta_{\nu c}}{D_S}-\frac{\delta_{\nu c}}{D_W}\right]\left[\left(\frac{T_H}{T_c}\right)^{(\beta-1)/2}-1\right]\right.\\&\quad\left.-c\frac{2\pi(L-H_S-H_W)}{\lambda_c}\left(\frac{\delta_{\nu c}}{D_W}\right)\right\}.\end{aligned} \quad (33)$$

From Eq. (33) it follows that, because of condition (2), the effective Reynolds stresses across the region $dT_m/dx\leqslant 0$ are negligible in comparison with the stresses across the stack. Thus, Eq. (33) can be rewritten in the form

$$\begin{aligned}S^{\mathrm{QA}}&=\oint s^{\mathrm{QA}}\,dx\\&\cong\frac{2\pi c|\tilde{p}|^2}{3\rho_ca_c^2}\left(\frac{\delta_{\nu c}}{D_W}\right)\left\{\frac{1}{2\pi}\left(1+\frac{d}{2c}\right)\left(\frac{\beta+1}{\beta-1}\right)\left(\frac{D_W}{D_S}\right)\right.\\&\quad\left.\times\left[\left(\frac{T_H}{T_c}\right)^{(\beta-1)/2}-1\right]-1\right\},\end{aligned}$$

where the approximation of Eq. (26) and also that $L\approx\lambda_c$ have been applied. Because typically[25,26] $|d/2c|\ll 1$ and $|(\beta-1)/2|\ll 1$ ($\propto 0.1$ in air), we can additionally simplify the expression as follows:

$$\begin{aligned}S^{\mathrm{QA}}&\equiv\oint s^{\mathrm{QA}}\,dx\\&\cong\frac{2\pi c|\tilde{p}|^2}{3\rho_ca_c^2}\left(\frac{\delta_{\nu c}}{D_W}\right)\left\{\frac{1}{2\pi}\left(\frac{\beta+1}{2}\right)\left(\frac{D_W}{D_S}\right)\ln\left(\frac{T_H}{T_c}\right)-1\right\}.\end{aligned} \quad (34)$$

In accordance with Eq. (34) the direction of the net stress difference (inducing the mass flow) depends, in general, on the stack heating. For example, it follows formally from Eq. (34) that the direction of the net stress difference in the case $T_H\rightarrow T_c$ is opposite to the direction of the amplified wave (and, consequently, $\overline{M}$ will be directed anticlockwise in Fig. 1). However, in typical thermoacoustic prime-movers, above the threshold the inequality $T_H\geqslant 2T_c$ holds. Then, because of condition (2) the net source of the mass flow is positive. It is controlled by the stack region, and can be approximated by

$$S^{QA} \cong \frac{(c+d/2)}{3\rho_c a_c^2}\left(\frac{\beta+1}{\beta-1}\right)\left(\frac{\delta_{\nu c}}{D_S}\right)^{QA} \times\left[\left(\frac{T_H}{T_c}\right)^{(\beta-1)/2}-1\right]|\tilde{p}|^2 \approx \frac{c(\beta+1)}{6\rho_c a_c^2}\left(\frac{\delta_{\nu c}}{D_S}\right)^{QA}\ln\left(\frac{T_H}{T_c}\right)|\tilde{p}|^2. \quad (35)$$

The superscript QA is used in Eq. (35) to denote that corresponding functions are evaluated in the quasiadiabatic regime. Note the weak (logarithmic) explicit dependence of the streaming source on the maximum normalized heating (T_H/T_c). However, we should not forget that there is also in Eq. (35) an implicit dependence on temperature, because the amplitude of the pressure oscillations depends on the external heating.

The denominator in Eq. (12), which plays a role of the total hydrodynamic resistance for the induced mass flow, can be rewritten in the form

$$\oint\left(\frac{2}{D}\right)^2\nu_m\,dx=\left(\frac{2}{D_S}\right)^2\nu_c\int_0^{H_S}T_N^{\beta+1}(x)dx \times\left\{1+\left(\frac{D_S}{D_W}\right)^2\frac{\int_{H_S}^{H_S+H_W}T_N^{\beta+1}(x)dx}{\int_0^{H_S}T_N^{\beta+1}(x)dx} +\left(\frac{D_S}{D_W}\right)^2\frac{L}{\int_0^{H_S}T_N^{\beta+1}(x)dx}\right\}. \quad (36)$$

Using a linear model for the temperature distribution in the regions $0\leqslant x\leqslant H_S$ and $H_S\leqslant x\leqslant H_S+H_W$, the second term in the curled brackets in Eq. (36) can be approximated as $\approx(D_S/D_W)^2(H_W/H_S)\ll(D_S/D_W)^2(L/H_S)$. This latter strong inequality follows from condition (6). Using the inequality $T_N\geqslant 1$ the third term in the curled brackets in Eq. (36) is estimated similarly as $\leqslant(D_S/D_W)^2(L/H_S)$. Consequently, because of the assumption (4) both the second and the third terms in Eq. (36) are negligible and hence we see that the dominant contribution to the total hydrodynamic resistance is provided by the stack since

$$\oint\left(\frac{2}{D}\right)^2\nu_m\,dx=\left(\frac{2}{D_S}\right)^2\nu_c\int_0^{H_S}T_N^{\beta+1}(x)dx. \quad (37)$$

Combining results (37) and (35), we derive the following solution for the average directional mass flow in the case of the quasiadiabatic stack:

$$\bar{M}^{QA}\cong\frac{(c+d/2)}{12\pi\rho_c a_c^3}\,\frac{\left(\frac{\beta+1}{\beta-1}\right)\left[\left(\frac{T_H}{T_c}\right)^{(\beta-1)/2}-1\right]}{\frac{1}{H}\int_0^{H_S}T_N^{\beta+1}(x)dx}\left(\frac{\lambda_c}{H_S}\right) \times\left(\frac{D_S}{\delta_{\nu c}}\right)|\tilde{p}|^2 \propto\frac{1}{12\pi\rho_c a_c^3}\left(\frac{\lambda_c}{H_S}\right)\left(\frac{D_S}{\delta_{\nu c}}\right)^{QA}|\tilde{p}|^2. \quad (38)$$

The latter rough, order-of-magnitude estimate is obtained under the assumptions $c\propto\beta\propto T_N\propto(\Delta T/T_c)\propto 1$. Equation (38), together with the solutions (22) and (30), provides in accordance with Eq. (14) a description of the streaming velocity. However, a comparison of $\bar{M}^{QA}$ [Eq. (38)] with $\langle\overline{\rho v_x}\rangle^{QA}$ both in the cold [Eq. (22)] and in the heated [Eq. (30)] parts of the annular prime-mover demonstrates that in the QA regime $\bar{M}^{QA}$ significantly exceeds $\langle\overline{\rho v_x}\rangle^{QA}$ everywhere in the device $[\bar{M}^{QA}/\langle\overline{\rho v_x}\rangle_c^{QA}\propto(1/6\pi)(L/H_S)(D_S/\delta_{\nu c})\gg 1$, $\bar{M}^{QA}/\langle\overline{\rho v_x}\rangle_h^{QA}\geqslant(D_S/\delta_{\nu c})^2\gg 1$, because of assumptions (3) and (19)]. In these latter estimates we have omitted for compactness the factors of the order of 1 contributed by the combinations of the dimensionless parameters c, β, γ, and σ (all of which are of the order of 1). Consequently, in the QA regime of the interaction between the acoustic and the thermal waves inside the stack, the velocity of the acoustic streaming can be approximated by

$$v_{xm}^{QA}\cong\bar{M}^{QA}/\rho_m. \quad (39)$$

The very rough order-of-magnitude estimate $\bar{v}_{xm}^{QA}\propto(1/12\pi)\times(\lambda_c/H_S)(D_S/\delta_{\nu c})^{QA}(|\tilde{v}_x|^2/a_c)$ which can be used for the streaming velocity follows from Eq. (39), Eq. (38), and the relation $|\tilde{p}|\approx a_c\rho_c|\tilde{v}_x|$ (approximately valid in the QA regime). This estimate mainly demonstrates the dependence of the streaming velocity on the acoustic thickness of the stack $[(\lambda_c/H_S)\gg 1$ because of the condition (26)] and on the parameter of the adiabaticity $[(D_S/\delta_{\nu c})\gg 1$ because of the assumption (19)] when the amplitude of the acoustic oscillations is fixed. As an example, we have estimated that, for the reported magnitude of the oscillating component of particle velocity $|\tilde{v}_x|\approx 3$ m/s in the annular prime-mover described in Ref. 18, this theoretical result predicts unidirectional streaming with a characteristic velocity $\bar{v}_{xm}^{QA}\approx 0.1$ m/s.

IV. QUASI-ISOTHERMAL REGIME

In the quasi-isothermal (QI) regime of the interaction between the acoustic waves and the surfaces of the stack (which is characterized by the fact that the width of the viscous and thermal boundary layers is significantly larger than the distance between the channel walls $\delta_{\nu,k}\gg D_S$), there is a further small parameter in the theory

$$(D_S/\delta_\nu)^2\ll 1. \quad (40)$$

Under condition (40) the functions $f_{\nu,k}$ [Eq. (16)] controlling the thermoacoustic interaction inside the stack can be approximated by

$$f_\nu\cong 1+i\frac{1}{6}\left(\frac{D_S}{\delta_\nu}\right)^2-\frac{1}{30}\left(\frac{D_S}{\delta_\nu}\right)^4-i\frac{17}{9\cdot 8\cdot 7\cdot 5}\left(\frac{D_S}{\delta_\nu}\right)^6+\cdots,$$
$$f_k\cong 1+i\frac{\sigma}{6}\left(\frac{D_S}{\delta_\nu}\right)^2-\frac{\sigma^2}{30}\left(\frac{D_S}{\delta_\nu}\right)^4-i\frac{17\sigma^3}{9\cdot 8\cdot 7\cdot 5}\left(\frac{D_S}{\delta_\nu}\right)^6+\cdots. \quad (41)$$

Retaining in Eq. (15) only the predominant terms important for the subsequent analysis, we may rewrite it in the form

$$\frac{d^2\tilde{p}}{dx^2}-(\beta+1)\frac{1}{T_N}\frac{dT_N}{dx}\frac{d\tilde{p}}{dx}+ik_c^2(6\gamma)\left(\frac{\delta_\nu}{D}\right)^2\frac{1}{T_N}\tilde{p}=0. \quad (42)$$

For the net amplification, the effects inside the stack caused by the presence of the temperature gradients should dominate over the effects of thermoviscous attenuation described by the last term in Eq. (42). The estimate of the pressure gradient inside the acoustically thin QI stack [which follows from the boundary condition (28) and the relation (23) in the cold part of the resonator] is

$$d\tilde{p}/dx \propto -6k_c(\delta_{\nu c}/D_S)^2\tilde{p}. \tag{43}$$

With the aid of Eq. (43), we can compare the second and the third term in Eq. (42) and we arrive at the conclusion that under condition (29) (where $H=H_S$) the wave equation inside the QI stack can be approximated by

$$\frac{d^2\tilde{p}}{dx^2}-(\beta+1)\frac{1}{T_N}\frac{dT_N}{dx}\frac{d\tilde{p}}{dx}\cong 0. \tag{44}$$

Under the assumption (29) of the dominance of the dT_N/dx-dependent effects, we neglect the terms $\propto(\omega/a)^2 \propto 1/\lambda^2$ in Eqs. (17) and (18) when evaluating $\langle\overline{\rho v_x}\rangle_h^{\mathrm{QI}}$ and s_h^{QI}. From Eq. (18) [using the expansions (41) to evaluate the functions presented in Appendix B], we obtain

$$\begin{aligned}\langle\overline{\rho v_x}\rangle_h^{\mathrm{QI}} &\cong \frac{17\sigma}{10\cdot 9\cdot 8\cdot 7}\frac{1}{\omega^3\rho_m}\left(\frac{D_S}{\delta_\nu}\right)^6\frac{1}{T_N}\frac{dT_N}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\\ &\cong \frac{17\sigma}{280}\frac{1}{\omega\rho_c a_c^2}\left(\frac{D_S}{\delta_\nu}\right)^2\frac{dT_N}{dx}|\tilde{p}|^2\\ &\propto \frac{17}{140}\frac{1}{\pi\rho_c a_c^2}\left(\frac{D_S}{\delta_{\nu c}}\right)^2\frac{\lambda_c}{H_S}|\tilde{p}|^2.\end{aligned} \tag{45}$$

The second form in Eq. (45) is obtained with the aid of Eq. (43). The third form in Eq. (45) is an order-of-magnitude estimate under the crude assumptions $\sigma\propto T_N\propto(\Delta T/T_c)\propto 1$. The solution for the density of the mass flow sources in the QI stack, obtained from an evaluation of Eq. (17), is

$$\begin{aligned}s_h^{\mathrm{QI}} \cong &-\frac{1}{\omega^2\rho_m}\frac{1}{9\cdot 8\cdot 7\cdot 5}\left(\frac{D_S}{\delta_\nu}\right)^4 \mathrm{Re}\left(\frac{1}{T_N}\frac{dT_N}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\right.\\ &\left.\times[9(\beta+1)-8\beta\sigma-17\sigma]-\frac{d^2\tilde{p}}{dx^2}\frac{d\tilde{p}^*}{dx}\right).\end{aligned}$$

Using Eqs. (44) and (43), we can transform this latter solution into the final form

$$\begin{aligned}s_h^{\mathrm{QI}} &\cong \frac{(8\beta+17)\sigma}{9\cdot 8\cdot 7\cdot 5}\frac{1}{\omega^2\rho_m}\left(\frac{D_S}{\delta_\nu}\right)^4\frac{1}{T_N}\frac{dT_N}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\\ &\cong \frac{(8\beta+17)\sigma}{70}\frac{1}{\rho_c a_c^2}\frac{dT_N}{dx}|\tilde{p}|^2.\end{aligned}$$

Consequently, the difference in the Reynolds stresses across the QI stack contributing to the total source of the mass flow in Eq. (12) can be found

$$S_h^{\mathrm{QI}} \equiv \int_0^{H_S} s_h^{\mathrm{QI}}dx \cong \frac{(8\beta+17)\sigma}{70}\frac{1}{\rho_c a_c^2}\frac{\Delta T}{T_c}|\tilde{p}|^2. \tag{46}$$

Using Eq. (33), the source of the mass flow in the inhomogeneously heated quasiadiabatic region $H_S \leqslant x \leqslant H_S+H_W$ is estimated to be roughly a factor of $(\delta_{\nu c}/D_W)^{\mathrm{QA}}$ smaller than the source described by Eq. (46) and, consequently, since this factor is $\ll 1$, this source of mass flow can be neglected. Then, the net Reynolds stresses inducing the mass flow can be presented in the form

$$\begin{aligned}S^{\mathrm{QI}} &\equiv \int_0^{H_S} s_h^{\mathrm{QI}}dx + \int_{H_S+H_W}^{L} s_c^{\mathrm{QA}}\,dx\\ &\cong \frac{2\pi c}{3\rho_c a_c^2}\left(\frac{\delta_{\nu c}}{D_W}\right)^{\mathrm{QA}}\left\{\frac{3(8\beta+17)\sigma}{140\pi c}\left(\frac{D_W}{\delta_{\nu c}}\right)^{\mathrm{QA}}\right.\\ &\quad\left.\times\left[\frac{T_H}{T_c}-1\right]-1\right\}|\tilde{p}|^2.\end{aligned} \tag{47}$$

It follows from Eq. (47) that the direction of the net source of the mass flow depends, in general, on the stack heating. However, in typical thermoacoustic prime-movers it is true that $(\Delta T/T_c) \geqslant 1$ above the threshold and, as a result [because of the inequality (19) $(\delta_{\nu c}/D_W)^{\mathrm{QA}} \ll 1$] the net source of the mass flow is approximately equal to the difference in the Reynolds stresses across the stack [Eq. (46)]. Consequently, the flow is directed clockwise. Substituting Eqs. (46) and (37) into Eq. (12), we derive the solution for the mass flow

$$\overline{M}^{\mathrm{QI}} \cong \frac{(8\beta+17)\sigma}{280\pi\rho_c a_c^3}\left(\frac{\lambda_c}{H_S}\right)\left[\left(\frac{D_S}{\delta_{\nu c}}\right)^2\right]^{\mathrm{QI}}\frac{\left(\frac{T_H}{T_c}-1\right)}{\frac{1}{H_S}\int_0^{H_S}T_N^{\beta+1}(x)dx}|\tilde{p}|^2. \tag{48}$$

A comparison of the solution (48) with solutions (22), (30), and (45) for $\langle\overline{\rho v_x}\rangle$ in various parts of the device demonstrates that the direction of the acoustic streaming velocity $\overline{v}_{xm}$ [Eq. (14)] can be different in the different parts of the resonator. It is always clockwise in the heated quasiadiabatic region $H_S \leqslant x \leqslant H_S+H_W$, because here, in accordance with Eq. (30), $\langle\overline{\rho v_x}\rangle$ is negative (while $\overline{M}^{\mathrm{QI}}$ is always positive). In the cold part of the resonator the direction of $\overline{v}_{xm}$ depends on the magnitude of the parameter $(\lambda_c/H_S)(D_S/\delta_{\nu c})^2$, which in turn is controlled by the inequalities (26) and (40). By substituting Eqs. (48) and (45) into Eq. (14), we find that inside the QI stack

$$\begin{aligned}\overline{v}_{xm}^{\mathrm{QI}} \cong &\frac{\sigma}{280\pi\rho_m\rho_c a_c^3}\left(\frac{\lambda_c}{H_S}\right)\left[\left(\frac{D_S}{\delta_{\nu c}}\right)^2\right]^{\mathrm{QI}}\\ &\times\frac{\left\{8\beta+17\left[1-\frac{1}{H_S}\int_0^{H_S}T_N^{\beta+1}(x)dx\right]\right\}}{\frac{1}{H_S}\int_0^{H_S}T_N^{\beta+1}(x)dx}\left(\frac{T_H}{T_c}-1\right)|\tilde{p}|^2.\end{aligned} \tag{49}$$

Note that the term in the square bracket in Eq. (49) is negative for $\beta > -1$, because the inequality $T_N \geqslant 1$ holds inside the stack. Consequently, both the magnitude and the direction of the velocity in Eq. (49) depend significantly on the temperature distribution inside the stack.

V. COMPARISON OF THE ACOUSTICALLY INDUCED AND STREAMING-INDUCED ENTHALPY FLOW

The solutions for the mass flow $\overline{M}$ [the general solution (12) and (17), and the solutions in particular limiting cases (38) and (48)] provide an opportunity to describe the associated enthalpy flow $\overline{M}c_pT_m$ in the resonator. Though, in principle, this enthalpy flow is induced originally by the acoustic waves we call it here the *streaming-induced* enthalpy flow in order to distinguish it from the enthalpy flow $\rho_m c_p\langle\overline{v_xT}\rangle$, for which we retain the term *acoustically induced* enthalpy flow.

Using the descriptions of the velocity and temperature oscillations presented in Appendix A [Eqs. (A1) and (A3)], the acoustically induced enthalpy flow can be presented in the form

$$\rho_m c_p\langle\overline{v_xT}\rangle=\frac{c_pT_m}{2\omega^3\rho_m}\left\{\frac{1}{T_m}\frac{dT_m}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\right.\times\mathrm{Im}\left(\overline{\frac{1}{\sigma-1}F_k-\frac{\sigma}{\sigma-1}F_\nu-F_\nu^*-\frac{1}{\sigma-1}F_kF_\nu^*}\right)-(\gamma-1)\left(\frac{\omega}{a}\right)^2\left.\times\mathrm{Im}\left[(\overline{1-F_k-F_\nu^*+F_kF_\nu^*})\tilde{p}\frac{d\tilde{p}^*}{dx}\right]\right\}. \quad (50)$$

With the aid of Eqs. (B7) and (B8) from Appendix B, we can present Eq. (50) in the form

$$\rho_m c_p\langle\overline{v_xT}\rangle=\frac{c_pT_m}{2\omega^3\rho_m}\left\{\frac{1}{\sigma^2-1}\frac{1}{T_m}\frac{dT_m}{dx}\left|\frac{d\tilde{p}}{dx}\right|^2\mathrm{Im}(f_k+\sigma f_\nu^*)\right.-(\gamma-1)\left(\frac{\omega}{a}\right)^2\mathrm{Im}\left[\left(1-\frac{1}{\sigma+1}f_k\right.\right.\left.\left.\left.-\frac{\sigma}{\sigma+1}f_\nu^*\right)\tilde{p}\frac{d\tilde{p}^*}{dx}\right]\right\}. \quad (51)$$

This solution coincides with one presented in Ref. 1 [Eq. (A30)] (for the case of an ideal gas and negligible temperature oscillations of the walls and stack).

In general, the expression for the acoustically induced enthalpy flow (51) should be compared with the solution for the streaming-induced enthalpy flow which follows from Eqs. (12) and (17). Here, to make the comparison of $\rho_m c_p\langle\overline{v_xT}\rangle$ and $\overline{M}c_pT_m$ for inside the thermoacoustic stack ($0\leqslant x\leqslant H_S$), we use the approach proposed in previous sections (Secs. III and IV). Applying condition (29), we retain in expression (51) only the terms induced by the inhomogeneous heating [i.e., we neglect the terms $\propto(\omega/a)^2\propto 1/\lambda^2$]. Then, a comparison of this approximate solution with expression (18) for $\langle\overline{\rho v_x}\rangle$ (similarly simplified) provides

$$\rho_m c_p\langle\overline{v_xT}\rangle\cong -c_pT_m\langle\overline{\rho v_x}\rangle. \quad (52)$$

Consequently, a comparison of the streaming-induced and of the acoustically induced enthalpy flows $\overline{M}c_pT_m$ and $\rho_m c_p\langle\overline{v_xT}\rangle$ can be obtained comparing the total mass flow $\overline{M}$ with its component $\langle\overline{\rho v_x}\rangle$. This has been done earlier in Secs. III and IV. Using the results in these sections and Eq. (52), we conclude that in the quasiadiabatic regime of the thermoacoustic interaction inside the stack

$$\left|\frac{\overline{M}c_pT_m}{\rho_m c_p\langle\overline{v_xT}\rangle}\right|^{\mathrm{QA}}\cong\left|\frac{\overline{M}^{\mathrm{QA}}}{\langle\overline{\rho v_x}\rangle_h^{\mathrm{QA}}}\right|\propto\left[\left(\frac{D_S}{\delta_{\nu c}}\right)^2\right]^{\mathrm{QA}}\gg 1. \quad (53)$$

Thus, in the QA regime the streaming-induced enthalpy flow significantly exceeds the acoustically induced enthalpy flow.

In the quasi-isothermal regime of the thermoacoustic interaction inside the stack

$$\left|\frac{\overline{M}c_pT_m}{\rho_m c_p\langle\overline{v_xT}\rangle}\right|^{\mathrm{QI}}\cong\left|\frac{\overline{M}^{\mathrm{QI}}}{\langle\overline{\rho v_x}\rangle_h^{\mathrm{QI}}}\right|\propto 1+\frac{8\beta}{17}\propto 1.$$

Consequently, in this QI regime the streaming-induced and the acoustically induced enthalpy flows are comparable in magnitude. More precise formulas comparing $\overline{M}c_pT_m$ and $\rho_m c_p\langle\overline{v_xT}\rangle$ can be obtained straightforwardly from the solutions derived in the previous sections, if required.

It is important to note that under the assumption of condition (29) the net enthalpy flow inside the thermoacoustic stack (i.e., the sum of the streaming-induced and the acoustically induced enthalpy flows) is controlled only by the hydrodynamic velocity $\overline{v}_{xm}$ of the acoustic streaming. In fact, with the aid of Eqs. (14) and (52) we obtain

$$\overline{M}c_pT_m+\rho_m c_p\langle\overline{v_xT}\rangle\cong c_pT_m(\overline{M}-\langle\overline{\rho v_x}\rangle)=c_pT_m\rho_m\overline{v}_{xm}=c_pT_c\rho_c\overline{v}_{xm}. \quad (54)$$

This result underlines the importance of the evaluation of the streaming velocity $\overline{v}_{xm}$ carried out in Secs. II, III, IV. Of particular significance, Eq. (49) demonstrates the dependence of the direction of the net enthalpy flow inside the QI stack on the heating of the stack.

VI. CONCLUSIONS

We have investigated the possibility of the generation of acoustic streaming in an annular thermoacoustic prime-mover. The theory developed predicts that an annular thermoacoustic prime-mover above the threshold of traveling wave excitation provides circulation of fluid (without using any moving parts and without an externally induced pressure gradient). The solutions show that the direction of the total cross-sectional average mass flow $\overline{M}$ and the direction of the cross-sectional average streaming velocity $\overline{v}_{xm}$ coincide with the direction of propagation of the amplified acoustic wave (clockwise in Fig. 1) in the case of the quasiadiabatic thermoacoustic interaction inside the stack. In the quasi-isothermal stack, the direction of $\overline{v}_{xm}$ depends on the level of stack heating and the parameter β [see Eq. (49)].

It is demonstrated that inside the stack during operation the net cross-sectional average flow of the enthalpy [i.e., the sum of the streaming-induced and acoustically induced enthalpy flows, Eq. (54)] is controlled by the streaming velocity $\overline{v}_{xm}$. In particular, the results of Secs. III and V [see Eqs. (39) and (53)] show that the enthalpy flow carried by the streaming significantly exceeds the enthalpy flux associated with the acoustically induced thermal conductivity of the gas. Thus, the acoustic streaming is predicted to play the dominant role in the enthalpy transport.

It should be noted that the analytical predictions obtained for the quasiadiabatic regime are currently the most interesting for applications, because they can also be used for order-of-magnitude estimates for the intermediate regime $D_S \geqslant \delta_{\nu,k}$ (which has been the most practical for thermoacoustic devices so far[1]). In fact, the lowest threshold for development of thermoacoustic instability in an annular thermoacoustic prime-mover was reported[18] to be for $D_S \approx 3\delta_k$. In Ref. 1, from both analytical and numerical analysis the conclusion is drawn that ''a practical engine can be expected... to have all available cross-sectional area filled with plates spaced from $2\delta_k$ to $4\delta_k$.''

Finally, it can be expected that the theory developed here will provide the necessary background for a theoretical modeling of a traveling wave thermoacoustic device having a quasi-isothermal stack as realized experimentally in Ref. 19. However, clearly, for this application the theory will require modification to take into account the additional acoustic elements in the setup of Ref. 19 that are not in the basic scheme (Fig. 1) we have considered.

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APPENDIX A: THE RELATIONS OF THE OSCILLATIONS OF THE PARTICLE VELOCITY, OF THE TEMPERATURE AND OF THE DENSITY IN THE ACOUSTIC FIELD WITH THE PRESSURE OSCILLATIONS

The components $\tilde{v}_x, \tilde{v}_y$ of the particle velocity, the complex amplitudes $\tilde{T}, \tilde{\rho}$ of the temperature, and the density variations can be presented as function of the pressure oscillations $\tilde{p}$ and its axial gradient $d\tilde{p}/dx$ as follows:[1,24]

$$\overline{v_x} = \frac{1}{i\omega\rho_m}[1-F_\nu]\frac{d\tilde{p}}{dx}, \tag{A1}$$

$$\tilde{v}_y = -\frac{1}{i\omega\rho_m}\left(\frac{D}{2}\right)\left\{\frac{d}{dx}\left[(\eta-\Phi_\nu)\frac{d\tilde{p}}{dx}\right] + \left(\eta - \frac{\sigma}{\sigma-1}\Phi_\nu + \frac{1}{\sigma-1}\Phi_k\right)\frac{1}{T_m}\frac{dT_m}{dx}\frac{d\tilde{p}}{dx} + \frac{\omega^2}{a^2}[\eta+(\gamma-1)\Phi_k]\tilde{p}\right\}, \tag{A2}$$

$$\tilde{T} = -\frac{1}{\omega^2\rho_m}\left[1-\frac{\sigma}{\sigma-1}F_\nu + \frac{1}{\sigma-1}F_k\right]\frac{dT_m}{dx}\frac{d\tilde{p}}{dx} + \frac{1}{\rho_m c_p}[1-F_k]\tilde{p}, \tag{A3}$$

$$\tilde{\rho} = -\frac{\rho_m}{T_m}\tilde{T} + \frac{\gamma}{a^2}\tilde{p} = \frac{1}{\omega^2}\left[1-\frac{\sigma}{\sigma-1}F_\nu + \frac{1}{\sigma-1}F_k\right] \times \frac{1}{T_m}\frac{dT_m}{dx}\frac{d\tilde{p}}{dx} + \frac{1}{a^2}[1+(\gamma-1)F_k]\tilde{p}. \tag{A4}$$

Here, $\rho \equiv \nu_m/k_m$ is the Prandtl number, a is the speed of sound (adiabatic), γ is the ratio of isobaric to isochoric specific heats. The functions $F_{\nu,k}$ and $\Phi_{\nu,k}$ which describe the transverse distribution of the acoustic field in the channels are defined by

$$F_{\nu,k} \equiv \frac{\cosh\left[\frac{D}{(1+i)\delta_{\nu,k}}\eta\right]}{\cosh\left[\frac{D}{(1+i)\delta_{\nu,k}}\right]}, \tag{A5}$$

$$\Phi_{\nu,k} \equiv \left[\frac{(1+i)\delta_{\nu,k}}{D}\right]\frac{\sinh\left[\frac{D}{(1+i)\delta_{\nu,k}}\eta\right]}{\cosh\left[\frac{D}{(1+i)\delta_{\nu,k}}\right]}, \tag{A6}$$

and, consequently,

$$F_{\nu,k} = \partial\Phi_{\nu,k}/\partial\eta. \tag{A7}$$

Note that the functions $f_{\nu,k}$ defined in Eq. (16) are particular values of the functions $\Phi_{\nu,k}$, i.e., $f_{\nu,k} = \Phi_{\nu,k}$ $(\eta=1)$. We remind here that $\eta \equiv 2y/D$.

APPENDIX B: SOME SPATIAL INTEGRALS OF THE FUNCTIONS CHARACTERIZING THE THERMOACOUSTIC INTERACTION

The following integrals contribute to solution (17) for the sources of the directional mass flow:

$$\overline{\int_{-1}^{\eta} d\eta'\ \Phi_{\nu,k}} = -\left[\frac{(1+i)\delta_{\nu,k}}{D}\right]^2(1-f_{\nu,k}), \tag{B1}$$

$$\overline{\int_{-1}^{\eta} d\eta'\ \eta' F_\nu} = 2\left[\frac{(1+i)\delta_\nu}{D}\right]^2(1-f_\nu) - f_\nu, \tag{B2}$$

$$\overline{\int_{-1}^{\eta} d\eta'\ \eta' F_\nu F_\nu^*} = f_\nu f_\nu^* - \frac{1}{2}(f_\nu + f_\nu^*) + \frac{1}{2}\left[\frac{(1+i)\delta_\nu}{D}\right]^2(f_\nu - f_\nu^*), \tag{B3}$$

$$\overline{\int_{-1}^{\eta} d\eta'\ \Phi_\nu F_\nu^*} = -\frac{1}{2}f_\nu f_\nu^* - \frac{1}{2}\left[\frac{(1+i)\delta_\nu}{D}\right]^2(1-f_\nu^*), \tag{B4}$$

$$\overline{\int_{-1}^{\eta} d\eta'\ \Phi_k F_\nu^*} = -\frac{1}{(\sigma+1)}f_k f_\nu^* - \frac{1}{(\sigma+1)}\left[\frac{(1+i)\delta_\nu}{D}\right]^2 \times \left(1-\frac{\sigma-1}{\sigma+1}f_k - \frac{2}{\sigma+1}f_\nu^*\right). \tag{B5}$$

Note that it is sufficient to assume in Eq. (B5) that $\sigma=1$ in order to get Eq. (B4). In order to get from Eq. (B5) the formula for

$$\overline{\int_{-1}^{\eta} d\eta' \, \Phi_\nu^* F_k},$$

it is sufficient to interchange $\nu \leftrightarrow k$ (and, consequently, σ should be replaced by $1/\sigma$) and take the complex conjugate

$$\overline{\int_{-1}^{\eta} d\eta' \, \Phi_\nu^* F_k} = -\frac{\sigma}{(\sigma+1)} f_k f_\nu^* + \frac{1}{(\sigma+1)} \times \left[\frac{(1+i)\delta_\nu}{D}\right]^2 \left(1+\frac{\sigma-1}{\sigma+1} f_\nu^* - \frac{2\sigma}{\sigma+1} f_k\right). \tag{B6}$$

The simplest integral contributing to Eq. (17) is

$$\overline{\int_{-1}^{\eta} \eta' \, d\eta'} = -\frac{1}{3}.$$

For an evaluation of the contribution of the term $(4\nu_m/D^2)\langle\overline{\rho v_x}\rangle$ to the mass flow source, we also use

$$\overline{F_k F_\nu^*} = \frac{\sigma}{\sigma+1} f_k + \frac{1}{\sigma+1} f_\nu^*, \tag{B7}$$

$$\overline{F_{\nu,k}} = f_{\nu,k}. \tag{B8}$$

[1] G. W. Swift, ''Thermoacoustic engines,'' J. Acoust. Soc. Am. **84**, 1145–1180 (1988).
[2] J. Wheatley, T. Hofler, G. W. Swift, and A. Migliori, ''Understanding of simple phenomena in thermoacoustics with applications to acoustical heat engines,'' Am. J. Phys. **53**, 147–162 (1985).
[3] A. A. Atchley, H. E. Bass, and T. J. Hofler, ''Development of nonlinear waves in a thermoacoustic prime mover,'' in *Frontiers in Nonlinear Acoustics: Proceedings of the 12th ISNA*, edited by M. F. Hamilton and D. T. Blackstock (Elsevier Science, London, 1990), pp. 603–608.
[4] G. Müller and W. Lauterborn, ''Experiments with thermoacoustic oscillator physical and musical,'' in *Proceedings of the International Symposium on Musical Acoustics* (Le Normont, Dourdan, France, 1995), pp. 178–183.
[5] V. A. Müller and E. Lang, ''Experimente mit thermisch getriebenen gasflüssigheits-schwingungen,'' J. Appl. Math. Phys. (ZAMP) **36**, 358–366 (1985).
[6] V. E. Gusev, H. Bailliet, P. Lotton, and M. Bruneau, ''Asymptotic theory of nonlinear acoustic waves in a thermoacoustic prime-mover,'' Acust. Acta Acust. **86**, 25–38 (2000).
[7] L. D. Landau, A. I. Akhiezer, and E. M. Lifshitz, *General Physics (Mechanics and Molecular Physics)* (Pergamon, Oxford, 1967).
[8] D. D. Hebert and A. A. Atchley, ''Measurements of the evolution of the temperature profile in a parallel plate stack,'' J. Acoust. Soc. Am. **100**, 2846 (1986).
[9] T. Yazaki, A. Tominauga, and Y. Narahara, ''Large heat transport due to spontaneous oscillation induced in a tube with steep temperature gradients,'' J. Heat Transfer **105**, 889–894 (1983).
[10] O. V. Rudenko and S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics* (Consultants Bureau, New York, 1977).
[11] N. Rott, ''The influence of heat conduction on acoustic streaming,'' J. Appl. Math. Phys. (ZAMP) **25**, 417–421 (1974).
[12] D. F. Gaitan, A. Gopinath, and A. A. Atchley, ''Experimental study of acoustic turbulence and streaming in a thermoacoustic stack,'' J. Acoust. Soc. Am. **96**, 3220(A) (1994).
[13] T. Yazaki and A. Tominaga, ''Measurement of sound generation in thermoacoustic oscillations,'' Proc. R. Soc. London, Ser. A **454**, 2113–2122 (1998).
[14] J. R. Olson and G. W. Swift, ''Acoustic streaming in pulse tube refrigerators: Tapered pulse tubes,'' Cryogenics **37**, 769–776 (1997).
[15] D. Gedeon, ''DC gas flows in Stirling and pulse-tube Cryocoolers,'' in *Cryocoolers*, edited by R. G. Ross (Plenum, New York, 1997), Vol. 9, pp. 385–392.
[16] G. W. Swift, D. L. Gardner, and S. Backhaus, ''Acoustic recovery of lost power in pulse tube refrigerators,'' J. Acoust. Soc. Am. **105**, 711–724 (1999).
[17] P. H. Ceperley, ''Gain and efficiency of a short traveling wave heat engine,'' J. Acoust. Soc. Am. **77**, 1239–1244 (1985), and references therein.
[18] T. Yazaki, A. Iwata, T. Maekawa, and A. Tominaga, ''Traveling wave thermoacoustic engine in a looped tube,'' Phys. Rev. Lett. **81**, 3128–3131 (1998).
[19] S. Backhaus and G. W. Swift, ''A thermoacoustic Stirling heat engine,'' Nature (London) **399**, 335–338 (1999).
[20] O. V. Rudenko and A. A. Sukhorukov, ''Nonstationary Eckart streaming and pumping of liquid in an ultrasonic field,'' Akust. Zh. **44**, 653–658 (1998) [Sov. Phys. Acoust. **44**, 565–570 (1998)].
[21] J. E. Piercy and J. Lamb, ''Acoustic streaming in liquids,'' Proc. R. Soc. London, Ser. A **226**, 43–50 (1954).
[22] L. M. Brekhovskikh and O. A. Godin, *Acoustics of Layered Media 1: Plane and Quasiplane Waves* (Springer, Berlin, 1991).
[23] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1982).
[24] N. Rott, ''Thermoacoustics,'' Adv. Appl. Mech. **20**, 135–175 (1980).
[25] N. Rott, ''Damped and thermally driven acoustic oscillations in wide and narrow tubes,'' Z. Angew. Math. Phys. **20**, 230–243 (1969).
[26] H. A. Kramers, ''Vibrations of a gas column,'' Physica (Amsterdam) **15**, 971–978 (1949).
[27] A. Larraza and A. A. Atchley, ''Analysis of a thermoacoustic prime mover above onset of self-oscillation,'' J. Acoust. Soc. Am. **94**, 1773 (1993).
[28] H. L. Kurtz, D. T. Blackstock, ''Attenuation of intense sinusoidal waves in air-saturated, bulk porous materials,'' J. Acoust. Soc. Am. **81**, 1723–1731 (1987).
[29] R. F. Lambert and J. D. Mcintosh, ''Nonlinear wave propagation through rigid porous materials. II. Approximate analytical solutions,'' J. Acoust. Soc. Am. **88**, 1950–1959 (1990).
[30] J. R. Olson and G. W. Swift, ''Energy dissipation in oscillating flow through straight and coiled pipes,'' J. Acoust. Soc. Am. **100**, 2123–2131 (1996).
[31] G. Mozurkewich, ''Time average temperature distribution in a thermoacoustic stack,'' J. Acoust. Soc. Am. **103**, 380–388 (1998).
[32] G. Mozurkewich, ''A model for transverse heat transfer in thermoacoustics,'' J. Acoust. Soc. Am. **103**, 3318–3326 (1998).
[33] V. E. Gusev and O. V. Rudenko, ''Nonsteady quasi-one-dimensional acoustic streaming in unbounded volumes with hydrodynamic nonlinearity,'' Akust. Zh. **25**, 875–881 (1979) [Sov. Phys. Acoust. **25**, 493–497 (1979)].